3d $N=4$ QFT and Ring Objects on the affine Grassmannian

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Introduction

Physicists assign Higgs and Coulomb branches ($\mathcal{M}_H(\mathcal{T})$, $\mathcal{M}_C(\mathcal{T})$)

to a 3d $\mathcal{N}=4$ SUSY QFT $\mathcal{T}$.

When the QFT $\mathcal{T}$ is a gauge theory (of cotangent type $\mathbf{M}=\mathbf{N}\oplus\mathbf{N}^*$),

— $\mathcal{M}_H(\mathcal{T})$ is a hyperKähler quotient $\mathbf{M}/\!/\!/G$,　　G : complexification of the gauge group

— $\mathcal{M}_C(\mathcal{T})$, as an affine algebraic variety, is defined mathematically rigorously by Braverman – Finkelberg – N in 2014.

<u>Question</u> : How about other 3d $\mathcal{N}=4$ QFT $\mathcal{T}$?
say 3d class S theories (Sicilian theories) ?

(This was asked by (Moore–)Tachikawa at String-Math 2011

• As a prize, I will offer a nice dinner at the Sushi restaurant in the University of Tokyo campus where the IPMU is.

A class S theory $\mathcal{T}$ is associated to
- $G^\vee$: cpx reductive group
- Σ : a Riemann surface possibly with punctures

It is obtaind as (a mysterious 6d theory)$/\Sigma \times S^1$.
Nevertheless physicists believe that a class S theory is well-defined.

Coulomb branch
$\mathcal{M}_C(\mathcal{T})$ = additive version of $G^\vee$-character variety on Σ
Moore-Tachikawa asked Higgs branch $\mathcal{M}_H(\mathcal{T})$.

This space, again as an affine algebraic variety, is constructed by
- Ginzburg-Kazhdan (not yet circulated publicly)
- BFN 1706.02122

In BFN construction, both $\mathcal{M}_C$ for gauge theories and $\mathcal{M}_H$ for class S

are defined through ring objects in equivariant derived Satake category,

$D_{G_\mathcal{O}}(Gr_G)$ $G_\mathcal{O}$-equiv. derived category of constructible sheaves on the affine Grassmannian Gr_G, equipped with convolution product.

Why ring objects?

- Ring objects are functorial version of coordinate rings of Higgs/Coulomb branches for 3d $N=4$ QFT with symmetry.

(If a theory has no symmetry, a ring object is nothing but the coordinate ring of Higgs/Coulomb branch.

Recall gauge theory = free theory $/\!/\!/ G$

$\uparrow$ gauging — integral $\int_{G\text{-connections}} DA \cdots\cdots$

$\Rightarrow$ Higgs branches are related as

$$\mathcal{M}_H(\text{gauge theory}) = \underbrace{\mathcal{M}_H(\text{free theory})}_{\mathbb{M}} /\!/\!/ G \;=\; \mathbb{M} /\!/\!/ G$$

But we do not have simple relation between $\mathcal{M}_C(\text{gauge theory})$ and $\mathcal{M}_C(\text{free theory})$.

Rather we have $\underbrace{\mathcal{M}_C(\text{gauge theory})}_{\substack{\parallel \\ \text{ring object for } \underline{\text{no}} \text{ symmetry}}} = H^*_{G_\mathcal{O}}(\mathrm{Gr}_G, \text{ring object for free theory})$.

Namely gauging $\rightsquigarrow H^*_{G_\mathcal{O}}(\mathrm{Gr}_G, ?)$ on ring objects

– Moreover ring objects are easier to construct than Higgs/Coulomb branches, as we have more operations on them.

In particular, Higgs branches of class S theories are defined via ring objects.

In fact, it is known (Benini-Tachikawa-Xie)

$$\text{class S for } (G^\vee, \underset{\substack{g \text{ genus} \\ n \text{ punctures}}}{\Sigma}) = T[G^\vee]^n \times \mathrm{Hyp}(\mathfrak{g}^\vee \otimes \mathfrak{g}^{\vee *}) /\!/\!/ G^\vee_{\mathrm{diag}}$$

(more precisely Coulomb gauging)

Our construction follows from this + "conjecture" formulated for arbitrary 3d $N=4$ QFT.

☆ For $\mathcal{M}_C$ (gauge theory) we used cohomology of a moduli space.

For $\mathcal{M}_H$ (class S) we just use cohomology itself, not a moduli space.

Quick Review of the mathematical definition of $\mathcal{M}_C$ for a gauge theory

G : complex reductive group

$\mathbb{M}$: symplectic representation of G $\quad$ Assume $= \mathbb{N} \oplus \mathbb{N}^*$

flavor symmetry $\quad G \triangleleft \tilde{G}$ s.t. $\mathbb{M}$ is a representation of $\tilde{G}$

$G_F = \tilde{G}/G$

gauge theory for $(G, \mathbb{M})$ $\xrightarrow[\text{by } G_F]{\text{gauging}}$ gauge theory for $(\tilde{G}, \mathbb{M})$

☆ Higgs branch

$$\mathcal{M}_H = \mathbb{M} /\!/\!/ G$$

☆ Coulomb branch [BFN]

$$\mathrm{Gr}_G := G_K / G_{\mathcal{O}} \text{ affine Grassmannian} \cong \left\{ (\mathcal{P}, \varphi) \,\middle|\, \begin{array}{l} \mathcal{P}: G\text{-bdles on } D \\ \mathcal{P}|_{D^\times} \underset{\varphi}{\simeq} D^\times \times G \end{array} \right\} \Big/ \text{isom}$$

$$\mathcal{R} := \{ [g, s] \in G_K \times_{G_{\mathcal{O}}} \mathbb{N}_{\mathcal{O}} \mid gs \in \mathbb{N}_{\mathcal{O}} \} = \left\{ (\mathcal{P}, \varphi, s) \,\middle|\, \begin{array}{l} s \in \Gamma(\mathcal{P} \times_G \mathbb{N}) \\ \varphi(s) \in \mathbb{N}_{\mathcal{O}} \end{array} \right\} \Big/ \text{isom}$$

$\curvearrowleft G_{\mathcal{O}}$

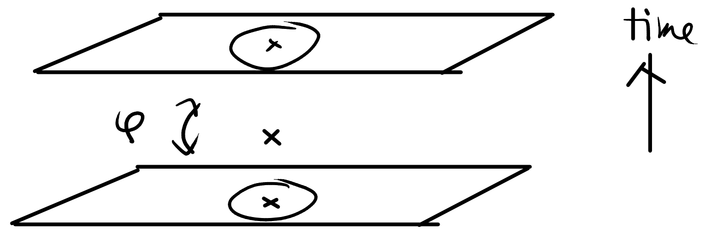
$G_K = G((z))$

$G_{\mathcal{O}} = G[[z]]$

$\mathbb{N}_{\mathcal{O}} = \mathbb{N}[[z]]$

The equivariant homology $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ has a convolution product, which is commutative.

Then $\quad \mathcal{M}_C := \operatorname{Spec} H_*^{G_{\mathcal{O}}}(\mathcal{R})$



Note that we have maps $\mathcal{R} \longrightarrow Gr_G \longrightarrow pt$

and $$H_*^{G_\mathcal{O}}(\mathcal{R}) = H^*_{G_\mathcal{O}}((\mathcal{R}\to pt)_* \omega_\mathcal{R})$$

dualizing complex on $\mathcal{R}$

This definition can be divided into two steps.

$$H^*_{G_\mathcal{O}}((Gr_G \to pt)_* \underbrace{(\mathcal{R}\to Gr_G)_* \omega_\mathcal{R}}_{= \mathcal{A}}) = H^*_{G_\mathcal{O}}(Gr_G, \mathcal{A})$$

$\mathcal{A} \in D_{G_\mathcal{O}}(Gr_G)$ (cohomology with coefficients in $\mathcal{A}$)

This $\mathcal{A}$ is an example of a ring object in the derived Satake category $D_{G_\mathcal{O}}(Gr_G)$

☆ flavor symmetry $\mathcal{R}_{\tilde{G},N} \longrightarrow Gr_{\tilde{G}} \longrightarrow Gr_{G_F} \overset{i}{\hookleftarrow} \{e\}$

$$H^*_{G_\mathcal{O}}(i^! (\mathcal{R}_{\tilde{G},N} \to Gr_{G_F})_* \omega_{\mathcal{R}_{\tilde{G},N}}) = \mathbb{C}[\mathcal{M}_C(G,N)]$$

gauging w.r.t. G_F

$$H^*_{G_\mathcal{O}}(Gr_{G_F}, (\mathcal{R}_{\tilde{G},N} \to Gr_{G_F})_* \omega_{\mathcal{R}_{\tilde{G},N}}) = \mathbb{C}[\mathcal{M}_C(\tilde{G},N)]$$

derived Satake category

G : complex reductive group $G_K = G((z))$
$Gr_G = G_K / G_{\mathcal{O}}$: affine Grassmannian $G_{\mathcal{O}} = G[[z]]$

Recall geometric Satake correspondence :

$$\underbrace{(\mathrm{Perv}_{G_{\mathcal{O}}}(Gr_G), *)}_{\text{Satake category}} \cong (\mathrm{Rep}\, G^\vee, \otimes)$$

equivalence of tensor categories

$\mathrm{Perv}_{G_{\mathcal{O}}}(Gr_G)$ = $G_{\mathcal{O}}$-equivariant perverse sheaves on Gr_G

$*$ = convolution product

$\mathrm{Rep}\, G^\vee$: finite dimensional representations of the Langlands dual group $G^\vee$

This is a very important result used in e.g. geometric Langlands conjecture.

It is also known that the convolution product is defined over a larger category : derived Satake category $D_{G_{\mathcal{O}}}(Gr_G)$

$G_{\mathcal{O}}$-equivariant derived category of constructible sheaves on Gr_G.
(more precisely ind-completion is required)

Def. A ring object is $\mathcal{A} \in D_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ equipped with

$$m : \mathcal{A} * \mathcal{A} \longrightarrow \mathcal{A}$$

such that ∘ associativity

$$\begin{tikzcd}
(\mathcal{A} * \mathcal{A}) * \mathcal{A} \arrow[r, phantom, "\cong"] \arrow[dr, "m \circ (m * \mathrm{id})"'] & \mathcal{A} * (\mathcal{A} * \mathcal{A}) \arrow[d, "m \circ (\mathrm{id} * m)"] \\
 & \mathcal{A}
\end{tikzcd}$$

∘ unit 1_{Gr_G} (= skyscraper sheaf at the origin) $1_{\mathrm{Gr}_G} \longrightarrow \mathcal{A}$ s.t.

$$\begin{tikzcd}
\mathcal{A} \cong \mathcal{A} * 1_{\mathrm{Gr}_G} \arrow[d] \arrow[dd, bend right, equal, "\mathrm{id}"'] \\
\mathcal{A} * \mathcal{A} \arrow[d, "m"] \\
\mathcal{A}
\end{tikzcd}$$

∘ commutativity

This can be loosely regarded as a family of vector spaces V_x parametrised by $x \in \mathrm{Gr}_G \simeq \Omega K$ (based loops for maximal compact subgroup K) together with homomorphisms $V_x \otimes V_y \longrightarrow V_{xy}$ + conditions

In particular, • $H^*_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathcal{A})$ are commutative rings.

• $!$-fiber at the identity $\in \mathrm{Gr}_G$

Another Fundamental example : Regular sheaf

$\mathcal{A}_R$ = regular sheaf $\xleftarrow{\text{geometric Satake}}$ $\mathbb{C}[G^\vee]$ regular representation of $G^\vee$

$$\mathbb{C}[G^\vee] \otimes \mathbb{C}[G^\vee] \xrightarrow[\text{mult.}]{} \mathbb{C}[G^\vee]$$

Arkhipov – Bezrukavnikov – Ginzburg (2004)

$H^*_{G_\mathcal{O}}(Gr_G, \mathcal{A}_R) = \mathbb{C}[G^\vee \times S^\vee]$ $S^\vee$ = Kostant-Slodowy slice for $G^\vee$

$!$-fiber at $1 = \mathbb{C}[N_{G^\vee}]$ nilpotent cone for $G^\vee$

Note $\mathcal{A}_R = \bigoplus_\lambda IC(Gr_G^\lambda) \otimes V_{G^\vee}(\lambda)^*$ as $\mathbb{C}[G^\vee] = \bigoplus_\lambda V_{G^\vee}(\lambda) \otimes V_{G^\vee}(\lambda)^*$

λ dominant coweight of G

Therefore $\mathcal{A}_R$ has an action of $G^\vee$

This is compatible with the $G^\vee$-action of $\begin{cases} G^\vee \times S^\vee \\ N_{G^\vee} \end{cases}$

More generally, there is a notion of H-equivariant ring objects in $D_{G_\mathcal{O}}(Gr_G)$ where H is another reductive group.

e.g. $\mathcal{A}_R$: $G^\vee$-equivariant ring object in $D_{G_\mathcal{O}}(Gr_G)$

3d class S theory

$\Sigma_{g,n}$: Riemann surface genus g, n punctures

$\mathcal{T} \equiv$ class S-theory for $(G^\vee, \Sigma_{g,n})$

$\mathcal{M}_H(\mathcal{T})$ depends only on g, n, as a cpx manifold (singularity in general)

$\mathcal{A}_R$: regular sheaf $\in D_{G_\mathcal{O}}(Gr_G)$

Define $\boxed{W_G^n := \operatorname{Spec} H^*_{G_\mathcal{O}}(Gr_G, \mathcal{A}_R^{\otimes n})}$

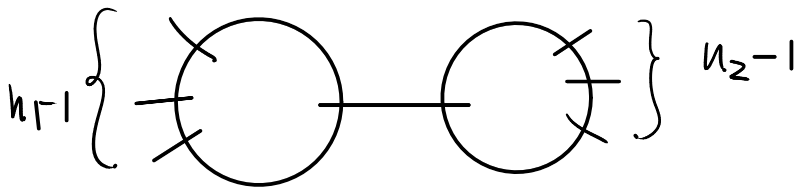
<u>Note</u> $\mathcal{A}_R^{\otimes n}$ is a ring object with $(G^\vee)^n$-symmetry

<u>Th</u>. [KG, BFN] W_G^n satisfies properties conjectured by Moore-Tachikawa:

- W_G^n is a Poisson variety (symplectic on smooth locus)
- It has $(G^\vee)^n$-hamiltonian action
- $W_G^1 = G^\vee \times S^\vee$, $W_G^2 = T^*G^\vee$

and gluing property:

$$W_G^{n_1} \times W_G^{n_2} /\!/\!/ \Delta G^\vee \cong W_G^{n_1+n_2-2}$$



Conjecture

Let us consider a 3d $\mathcal{N}=4$ QFT $\mathcal{T}$. Recall that we have notion of symmetry and gauging. In fact, we have two kinds, for Higgs and Coulomb.

Suppose $\mathcal{T}$ has
- G_H : Higgs symmetry.
- G_C : Coulomb symmetry

$$\Rightarrow \quad \mathcal{M}_H(\mathcal{T}) \curvearrowleft G_H \quad , \quad \mathcal{M}_C(\mathcal{T}) \curvearrowleft G_C \qquad \text{Hamiltonian}$$

and
$$\mathcal{M}_H(\underbrace{\mathcal{T} /\!\!/\!\!/ G_H}_{\text{Higgs gauging}}) = \mathcal{M}_H(\mathcal{T}) /\!\!/\!\!/ G_H$$

$$\mathcal{M}_C(\underbrace{\mathcal{T} /\!\!/\!\!/ G_C}_{\text{Coulomb gauging}}) = \mathcal{M}_C(\mathcal{T}) /\!\!/\!\!/ G_C$$

How about $\mathcal{M}_H(\mathcal{T} /\!\!/\!\!/_C G_C)$ and $\mathcal{M}_C(\mathcal{T} /\!\!/\!\!/_H G_H)$?

$\rightsquigarrow$ Need ring objects

Conjecture

A 3d $\mathcal{N}=4$ SUSY QFT $\mathcal{T}$ with
- G_H : Higgs symmetry
- G_C : Coulomb symmetry

$\overset{?}{\rightsquigarrow}$ Higgs branch ring object $\mathcal{A}_{\mathrm{Higgs}}(\mathcal{T}) \in D_{G_{C,\mathcal{O}}}(Gr_{G_C})$ with G_H-symmetry

and Coulomb branch ring object $\mathcal{A}_{\mathrm{Coulomb}}(\mathcal{T}) \in D_{G_{H,\mathcal{O}}}(Gr_{G_H})$ with G_C-symmetry

s.t. we have the "functorial" property w.r.t. gauging

- $\mathcal{M}_H(\mathcal{T} \underset{\mathrm{Coulomb}}{/\!/\!/} G_C) = \mathrm{Spec}\ H^*_{G_{C,\mathcal{O}}}(Gr_{G_C}, \mathcal{A}_{\mathrm{Higgs}}(\mathcal{T}))$

  $\mathcal{M}_C(\mathcal{T} \underset{\mathrm{Higgs}}{/\!/\!/} G_H) = \mathrm{Spec}\ H^*_{G_{H,\mathcal{O}}}(Gr_{G_H}, \mathcal{A}_{\mathrm{Coulomb}}(\mathcal{T}))$

Remark.

- $\mathcal{A}_{\mathrm{Higgs}}(\mathcal{T} \underset{\mathrm{Higgs}}{/\!/\!/} G_H) = \mathcal{A}_{\mathrm{Higgs}}(\mathcal{T}) /\!/\!/ G_H$, $\mathcal{A}_{\mathrm{Coulomb}}(\mathcal{T} \underset{\mathrm{Coulomb}}{/\!/\!/} G_C) = \mathcal{A}_{\mathrm{Coulomb}}(\mathcal{T}) /\!/\!/ G_C$

  (fiberwise hamiltonian reduction)

- I am happy to offer a sushi dinner for a solution.